INTRODUCTION TO ÉTALE COHOMOLOGY

SEMINAR SERIES ON PERVERSE SHEAVES
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Motivation

In the first seminar of the series we talked about a "Weil cohomology formalism", and now we would like to find a concrete realization of this theory.

As a first attempt, one might try to reason as follows: our variety (or scheme) X is a topological space with its Zariski topology, so we can consider the constant sheaf $\underline{\mathbb{Q}}$ on X and its cohomology groups

$$H^*(X, \underline{\mathbb{Q}}) .$$

However, there are some problems with this. One of these is that the Zariski topology is simply inadequate: indeed any constant sheaf $\underline{A}$ on an irreducible variety X is *flasque* (or *flabby*), which implies that

$$H^q(X, \underline{A}) = 0 \qquad \text{for all } i > 0 .$$

So we need to find a replacement for the Zariski topology. When X is a variety over $\mathbb{C}$ for example, we might want to use its (complex) analytic topology. However we are working over (the algebraic closure of) a finite field, and there seems to be no appropriate notion of topology on X in this case.

Grothendieck's idea to solve this problem was to give a new notion of "topology", in which open subsets are replaced by certain maps to X. This notion generalizes that of open immersions $U \hookrightarrow X$, and makes it possible to define sheaves over this new "topological space". Once we have have sheaves, we can simply define cohomology as the right derived functor of the global sections.

Another problem is given by the fact that $\mathbb{Q}$ is not a suitable coefficient field; we will explain this towards the end of this seminar.

1. Sheaves on a site

Let X be a topological space. Denote by $\mathrm{Open}(X)$ the category whose objects are open subsets of X and whose morphisms are defined, for $U, V \subseteq X$ open, as

$$\mathrm{Hom}(U,V) = \begin{cases} \{i_{UV}\} & \text{if } U \subseteq V \\ \emptyset & \text{otherwise} \end{cases}$$

where $i_{UV} : U \hookrightarrow V$ is the inclusion.

Then a presheaf of abelian groups on X is simply a contravariant functor F from $\mathrm{Open}(X)$ to the category $\underline{\mathrm{Ab}}$ of abelian groups. This means in particular that for each open $U \subseteq X$ we have an abelian group $F(U)$ and for each open $V \subseteq U$ we have restriction maps $\rho_V^U := F(i_{UV}) : F(U) \to F(V)$, compatible in the usual sense. The elements of $F(U)$ are also called *sections* of F on U, and we will simply denote the restriction of $s \in F(U)$ by $s|_V$ when there is no risk of confusion.

We can then define a sheaf on X as follows:

Definition 1.1. A presheaf (of abelian groups) F on X is called a *sheaf (of abelian groups)* on X if, for every $U \in \mathrm{Open}(X)$ and every open cover $\mathcal{U}$ of U, the sequence:

$$\begin{array}{cccccc} 0 & \to F(U) & \to & \prod_{V \in \mathcal{U}} F(V) & \to & \prod_{V,W \in \mathcal{U}} F(V \cap W) \\ & s & \mapsto & (s|_V)_{V \in \mathcal{U}} & & \\ & & & (s_V)_{V\in\mathcal{U}} & \mapsto & (s_V|_{V\cap W} - s_W|_{V \cap W})_{V,W\in\mathcal{U}} \end{array} \tag{1}$$

is exact.

This definition is perhaps elegant, but definitely cryptic at first sight. Lets break down what the exactness of the sequence (1) means in more concrete terms.

The injectivity of the map on the left means: a non-zero section restricts to the non-zero section at least in some open subset. In other terms, if you have two sections whose restrictions on every open subset coincide, then the two sections must be equal. This is known as the *identity* axiom.

Exactness in the middle means two things, the non-trivial one being: if you have a collection of sections $s_V \in V$ for every V in an open cover $\mathcal{U}$ of U, and these sections are compatible in the sense that they coincide on the intersections $V \cap W$, then there is a section $s \in U$ such that $s|_V = s_V$ for every $V \in \mathcal{U}$. This is known as the *gluing* axiom, because it allows us to glue together compatible sections to form one defined on a larger open set.

From this definition we can clearly see which particular feature of topological spaces is necessary in order to define sheaves: it is the notion of *open cover*. Grothendieck's idea was then to define a "topology" to be a certain category T with prescribed sets of maps to a given object as "open covers" of that object.

Definition 1.2 ([5, Chapter I, Definition 1.2.1])**.** A *site* (or Grothendieck topology) on a category T is given by specifying for every $U \in T$ which collection of maps to U are *covering families* for U, in such a way that:

(1) If $\{U_i \to U\}$ is a covering family for U and $V \to U$ is a morphism, then $\{U_i \times_U V \to V\}$ is a covering family for V.

(2) If $\{U_i \to U\}$ is a covering family for U and, for every U_i, $\{V_{ij} \to U_i\}$ is a covering family for U_i, then $\{V_{ij} \to U\}$, where each map is obtained by composition with $U_i \to U$, is a covering family for U.
(3) If $\varphi : V \to U$ is an isomorphism, then $\{\varphi\}$ is a covering family for U.

Compare this with the case $T = \mathrm{Open}(X)$: the axiom (T1) tells us that a subset of U is covered by a covering of U, and (T2) says that the union of open covers of an open cover is again an open cover.

Definition 1.3 ([5, Chapter I, Definition 1.2.3])**.** A *sheaf (of abelian groups)* on a site T is a contravariant functor $F : T \to \underline{\mathrm{Ab}}$ such that, for every $U \in T$ and every covering family $\mathcal{U} = \{i : U_i \to U\}$ of U, the following sequence is exact:

$$
\begin{array}{cccccc}
0 & \to F(U) & \to & \prod_{i \in \mathcal{U}} F(U_i) & \to & \prod_{i,j \in \mathcal{U}} F(U_i \times_U U_j) \\
 & s & \mapsto & (F(i)(s))_{i \in \mathcal{U}} & & \\
 & & & (s_i)_{i \in \mathcal{U}} & \mapsto & (F(j \times_U U_i)(s_i) - F(i \times_U U_j)(s_j))_{i,j \in \mathcal{U}}
\end{array}
\tag{2}
$$

Remark 1.4. The cumbersome notation "$F(j \times_U U_i)(s_i)$" can be explained looking at the following pullback diagram

$$
\begin{array}{ccc}
U_i \times_U U_j & \xrightarrow{i \times_U U_j} & U_j \\
{\scriptstyle j \times_U U_i}\downarrow & & \downarrow{\scriptstyle j} \\
U_i & \xrightarrow{\quad i \quad} & U
\end{array}
$$

and remembering that "$s_V|_{V \cap W}$" is just a shorthand for $F(\iota)(s_V)$ where $\iota : V \cap W \hookrightarrow V$ is the inclusion.

2. Étale morphisms

In this section we will assume that schemes are locally Noetherian and that morphisms of schemes are of finite type, see [2, 2.3.4 and 3.2.1]. These assumptions always hold for algebraic varieties and morphisms between them.

Definition 2.1. Let $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ be local rings. A ring homomorphism $f : A \to B$ is called *unramified* if $f(\mathfrak{m}_A) \cdot B = \mathfrak{m}_B$ and $A/\mathfrak{m}_A \hookrightarrow B/\mathfrak{m}_B$ is a separable field extension.

As an example, let $K \subseteq L$ be number fields with rings of integers $\mathcal{O}_K \subseteq \mathcal{O}_L$. A prime $\mathfrak{p}$ of K is unramified in L precisely when for every prime $\mathfrak{q}$ of L above $\mathfrak{p}$ the extension of local rings $\mathcal{O}_{K,\mathfrak{p}} \hookrightarrow \mathcal{O}_{L,\mathfrak{q}}$ is unramified in the sense of Definition 2.1.

Recall that, if A is a ring, an A-module M is called *flat* if for every injective A-homomorphism $N \hookrightarrow P$ the homomorphism $N \otimes_A M \to P \otimes_A M$ is also injective. It is a local property: a module M is flat if and only if $M_\mathfrak{p}$ is flat for every prime ideal $\mathfrak{p}$ of A.

We say that a morphism of rings $A \to B$ is flat if B is a flat A-module.

Definition 2.2. A morphism of schemes $f : X \to Y$ is called *étale at* $x \in X$ if the induced map of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat and unramified. We say that f is *étale* if it is so at every point of X.

Examples and facts.

(1) Open immersions are étale. This implies that the étale topology (see below) is a *refinement* of the Zariski topology.
(2) [1, §1, Proposition 1.7] A morphism of affine algebraic varieties over $\mathbb{C}$ is étale precisely when it is a locally biholomorphic map in the sense of complex analysis.
(3) [2, Theorem 4.3.12] If $f : X \to Y$ is a flat morphism (i.e. the induced maps of local rings are flat) then for every $y \in Y$ and every $x \in f^{-1}(y)$ we have $\dim_x f^{-1}(y) = \dim_x X - \dim_y Y$. If X is irreducible this simply means that the fibers of f all have dimension $\dim X - \dim Y$. If moreover f is étale, then $\dim X = \dim Y$ and every fiber is a finite set.
(4) A normalization morphism is never flat, thus never étale, unless it is an isomorphism, see [2, §8.1.4]). See for example [3, Theorem 23.0(i)]; the proof uses cohomological algebra, in particular Serre's normality criterion.
(5) It follows directly from (3) (and it is strictly related to (4)) that a blowup morphism is never flat.

Proposition 2.3 ([5, Chapter II, Proposition 1.1.2])**.**

(i) The composition of étale morphisms is étale.
(ii) If $f : X \to X'$ and $g : Y \to Y'$ are étale morphisms over a base scheme S, then so is $f \times_S g : X \times_S Y \to X' \times_S Y'$.
(iii) If $f : X \to Y$ and $g : Y \to Z$ are morphisms and both g and $g \circ f$ are étale, then f is étale.

Proof. Tamme [5] does not include a proof and just refers to EGA IV, so I will cite my personal favorite algebraic geometry reference instead.

Notice that (i) and (ii) can be rephrased as "étale morphisms are stable under composition" and "étale morphisms are stable under fibered products", respectively. These are contained in [2, Proposition 4.3.22(c)]. Part (iii) follows from the fact that étale morphisms are stable under base change, which is also part of [2, Proposition 4.3.22(c)] (base changing $g \circ f : X \to Z$ to $X \times_Z Y \to Z \times_Z Y$); see also [1, Chapter I, Remark 2.2]. □

3. Étale cohomology

Thanks to Proposition 2.3 one can easily show that the following definition satisfies all the axioms of a Grothendieck topology.

Definition 3.1. Let X be a variety and let $ET(X)$ be the category of étale schemes over X, i.e. the category whose objects are morphisms of schemes $Y \to X$ and whose objects are morphism $Y \to Y'$ that commute with the morphisms to X. Declaring as covering families all families of morphisms $\{\varphi : U_i \to U\}$ such that $\bigcup_i \varphi_i(U_i) = U$ we obtain a Grothendieck topology on $ET(X)$ which we call the *étale site* of X.

We may now consider sheaves on the étale site of X as defined in Definition 1.3. Following [5, Chapter I, §3.3], and as is the case with classical cohomology theory for sheaves on a topological space, for every sheaf F on X the global sections functor

$$F \mapsto \Gamma(X, F) = F(X)$$

is additive and left exact. The right-derived functor of $\Gamma(X, -)$ is called the *étale cohomology* of X with coefficients in F, and we will denote it by $H^*(X, F)$. If A is an abelian group and F is the constant sheaf $\underline{A}$ with value A, we will use the simpler notation $H^*(X, A)$.

Remark 3.2. The Čech cohomology groups $\check{H}^*(X, F)$ of a sheaf F on (any site of) a scheme X are defined as follows (but see [5, Chapter I, §2.2] or [2, §5.2] for a proper introduction to Čech cohomology). For any covering family $\{U_i \to X\}_{i \in I}$ of X and $n \geqslant 0$ we call

$$C^n(\{U_i \to X\}_{i\in I}, F) = \prod_{(i_0,\dots,i_n)\in I^{n+1}} F(U_{i_0} \times_X \cdots \times_X U_{i_n})$$

the group of n-cochains of $\{U_i \to X\}$ with values in F. We may define a coboundary operator

$$d^n : C^n(\{U_i \to X\}, F) \to C^{n+1}(\{U_i \to X\}, F)$$

by letting, for every $s \in C^n(\{U_i \to X\}, F)$, the $(i_0, \dots, i_{n+1})$-th component of $d^n s$ be

$$(d^n s)_{i_0,\dots,i_{n+1}} := \sum_{k=0}^{n+1} (-1)^k F(\pi_k)(s_{i_0,\dots,\widehat{i_k},\dots i_{n+1}})$$

where

$$\pi_k : U_{i_0} \times_X \cdots \times_X U_{i_{n+1}} \to U_{i_0} \times_X \cdots \times_X \widehat{U_{i_k}} \times_X \cdots \times_X U_{i_{n+1}}$$

is the projection. Then we have a cochain complex

$$C^0(\{U_i \to X\}, F) \to C^1(\{U_i \to X\}, F) \to \cdots \to C^n(\{U_i \to X\}, F) \to \dots$$

and we define Čech cohomology *of the covering* $\{U_i \to X\}$ as the cohomology of this complex

$$\check{H}^n(\{U_i \to X\}, F) := H^n(C^*(\{U_i \to X\}, F))$$

or more explicitly

$$\check{H}^n(\{U_i \to X\}, F) = \ker(d^n)/\operatorname{im}(d^{n-1}).$$

The Čech cohomology of X is then defined by taking the direct limit of these cohomology groups over all covering families of X

$$\check{H}^n(X, F) := \varinjlim_{\{U_i \to X\}} H^n(\{U_i \to X\}, F)$$

a map of coverings (also called *refinement*) being defined as a map between the set of indeces $\varepsilon : I \to I'$ together with maps $U_i \to U'_{\varepsilon(i)}$ that commute with the maps to X (see [5, Chapter I, Definition 2.2.4].

It turns out that, under certain assumptions, we may compute étale cohomology groups using Čech cohomology. Indeed, for any sheaf F on the étale site of X we have canonical group homomorphisms

$$\check{H}^n(X, F) \to H^n(X, F)$$

which are isomorphisms for $n = 0, 1$ and injective for $n = 2$ (see [5, Chapter I, Corollary 3.4.7]). Under certain conditions which hold for example when X is a quasi-projective variety, these maps are isomorphisms for every n (see [4, Chapter III, Theorem 2.17]).

In order to obtain a "Weil cohomology theory" in the sense of the previous talk, we need to take coefficients in a field K of characteristic 0 (that is, our sheaf F is the constant sheaf with value K). As mentioned in the introduction, one cannot hope that choosing $K = \mathbb{Q}$ works: if for example X is an elliptic curve one can show using the Künneth Formula that $\mathrm{End}(X)$ acts on $H^1(X, \mathbb{Q})$ compatibly with addition. By our axioms for the Weil cohomology formalism we need $H^1(X, \mathbb{Q})$ to be a dimension 2 vector space over $\mathbb{Q}$. This seems to be all nice and well until one remembers that if X is supersingular then $\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a central simple algebra of dimension 4 over $\mathbb{Q}$, and the aforementioned action of $\mathrm{End}(X)$ yields a representation

$$\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Mat}_{2\times 2}(\mathbb{Q})$$

which cannot exist, because the two unitary $\mathbb{Q}$ algebras above have the same (finite) dimension, but they are not isomorphic and the one on the left is simple.

One might still hope that choosing $K = \mathbb{Q}_p$ for some prime p works. Unfortunately this won't work either (see [1, Introduction to Chapter I]).

References

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