# The Local-Global Principle

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28 March 2019

Figure: Diophantus of Alexandria (III century)

# Diophantine Equations

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- "Fermat's Last Theorem" took centuries to be proved (stated in 1637 - proved to be true in 1995)

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In general, we have a **polynomial** with **integer coefficients**

$$F(X_1, \dots, X_n)$$

and we want to find **integer** solutions of

$$F(X_1, \dots, X_n) = 0.$$

# Modular Arithmetic

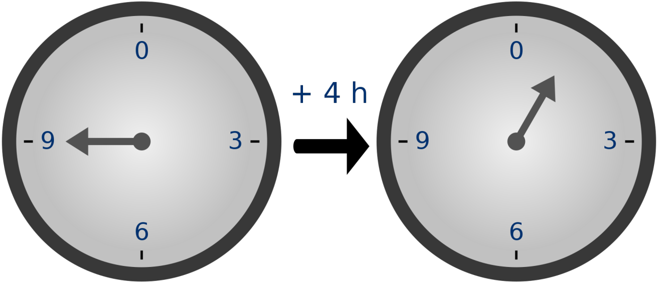


Figure: $9 + 4 = 1!$

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**Notation:** $$23 \equiv 11 \pmod{12}$$

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On the set $\{[0]_N, [1]_N, \dots, [N-1]_N\}$ we define operations:

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These operations behave well (associativity, commutativity...) and the set $\mathbb{Z}/N = \{[0]_N, [1]_N, \dots, [N-1]_N\}$ is a **ring**.

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Answer: 1h00!

- **Example 2.** (24-hour clock) If it is 17:00, what time was it 72 hours ago?

$$[17]_{24} - [72]_{24} = [17]_{24} - [0]_{24} = [17]_{24}$$

Answer: again 17:00!

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- But we can rewrite:

$$[X]_N^2 + [5]_N \cdot [Y]_N^2 = [10]_N \cdot [Z]_N^3 + [3]_N$$

And get an equation to be solved in $\mathbb{Z}/N$.

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- Since $\mathbb{Z}/5$ is a **finite set**, we can try all its elements to check if there is a solution.

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This means that there is no solution for the original equation!

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But one can check that it has no solution modulo 3.

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- **Remark:** The condition **(*)** is equivalent to the existence of a solution modulo every power of every prime number (**Chinese Remainder Theorem**).
- *Nitpicking note: together with* **(*)** *one should also assumes that the equation has solutions in the real numbers* $\mathbb{R}$.

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- But in some cases this idea works...

**Theorem (Hasse-Minkowski)**

*Let* $F(X_1, \ldots, X_n)$ *be a* **homogeneous** *polynomial of* **degree 2**. *If* $F(X_1, \ldots, X_n) = 0$ *has a solution modulo* $N$ *for every* $N$ *and it has a solution in* $\mathbb{R}$, *then it has an integer solution.*

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That is, **quadratic forms satisfy the local-global principle**.

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  **Example:** cubic forms **do not** satisfy it.
- Study **cohomological obstructions** to the principle (Brauer-Manin, descent).
- Apply the principle to study other number-theoretic problems.
- In my master thesis, I explained the lack of primitive solutions to quadratic equations with a Brauer-Manin obstruction to the local-global principle applied to “strong approximation”.

# More pictures

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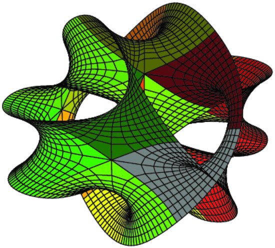

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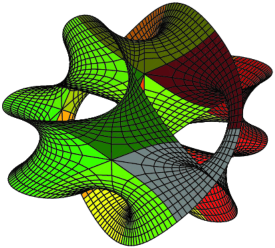

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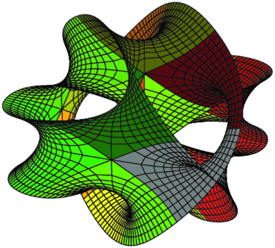


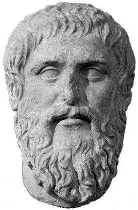

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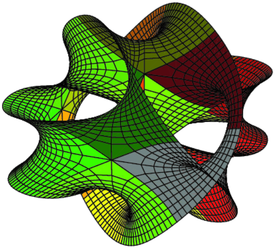


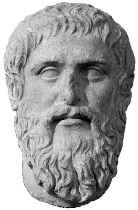

# Thank you for your attention!