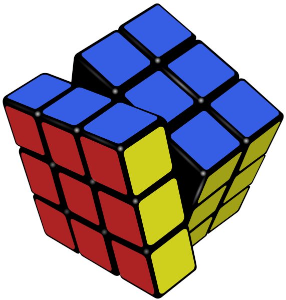
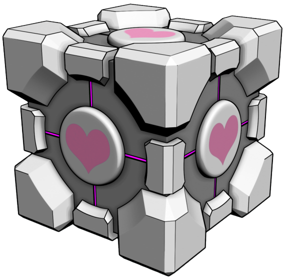
A tour of group theory with our companion cube

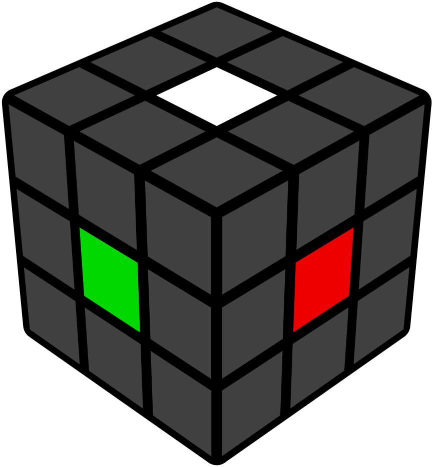
Sebastiano Tronto



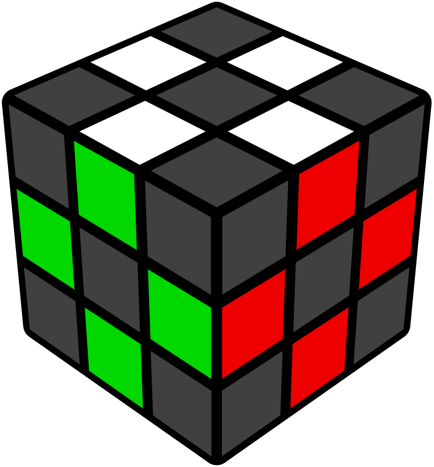


The cube

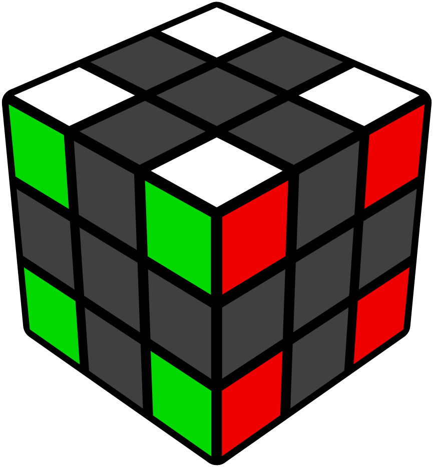
Centers



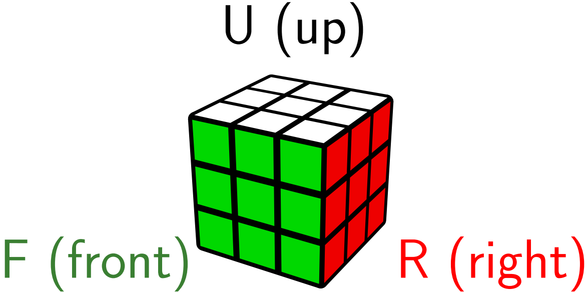
Edges



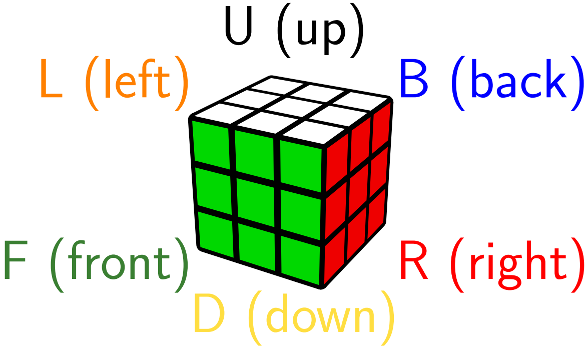
Corners



Moves



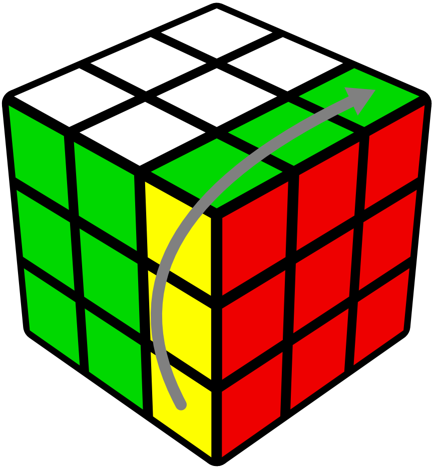
Moves



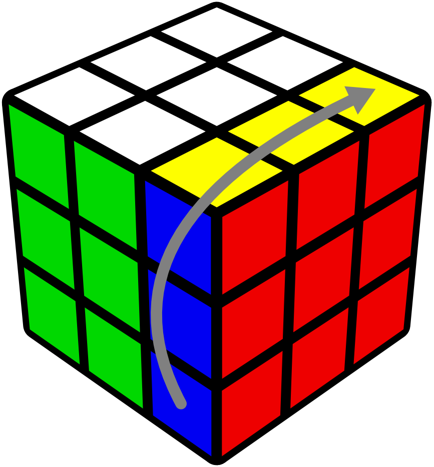
Moves

- Single letter: 90° clockwise (looking at that face)
- Repeated moves: $UUU\ldots = U^n$
- Sequences: R^2U: first do R^2, then U

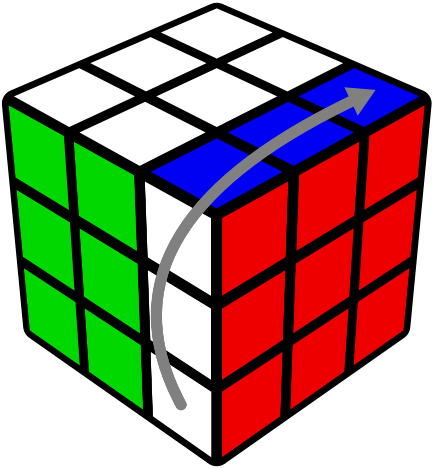
Example: R



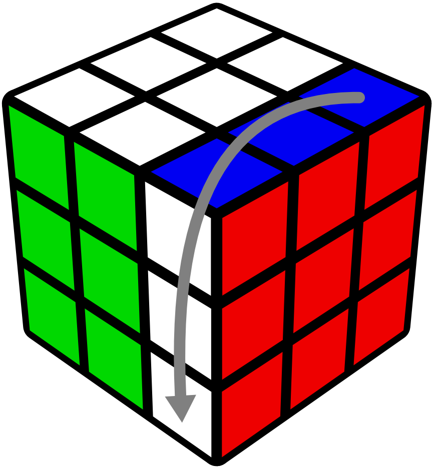
Example: R^2



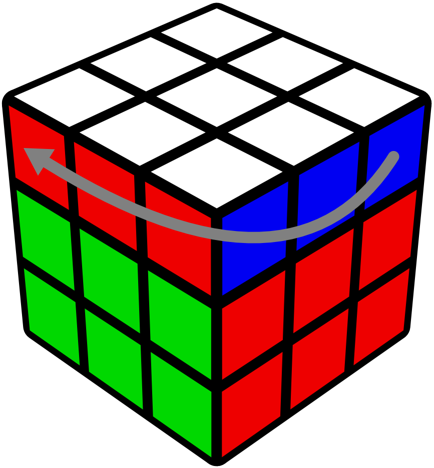
Example: R^3



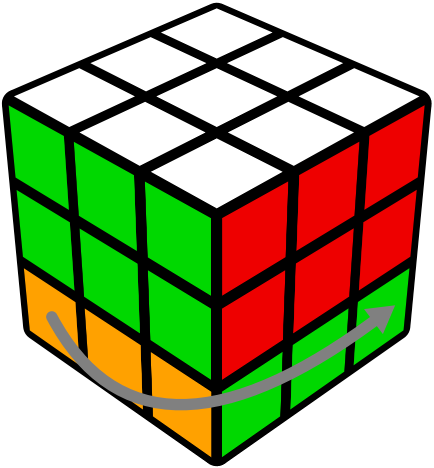
Example: R^3 vs R^{-1}



Example: U



Example: D



Groups

Definition

A group is a set G with an an operation $\times$ such that

- The operation $\times$ is associative
- There is a neutral element with respect to $\times$
- Every element has an inverse with respect to $\times$

Move sequences

$G = \{\text{sequences of moves on the cube}\}$
$\times =$ concatenation *and grouping exponents* $(U^2R \times R^{-1}U = U^3)$

- Associative: trivial
- Neutral element: empty sequence
- Inverse: e.g. the inverse of R^2UD^5F is $F^{-1}D^{-5}U^{-1}R^{-2}$

Free groups

Let $\mathcal{A}$ be a set and $\mathcal{A}^{-1} = \{a^{-1} \mid a \in \mathcal{A}\}$ (just symbols)

Definition

The *free group* on $\mathcal{A}$ is given by

- $G = \{$sequences of elements of $\mathcal{A} \cup \mathcal{A}^{-1}\}$
- $\times =$ concatenation and grouping exponents

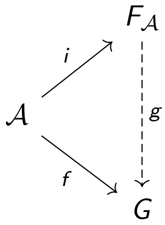
Free groups: examples

- $\mathbb{Z}$: free group on $\mathcal{A} = \{*\}$
- Moves on the cube: free group on $\mathcal{A} = \{U, D, R, L, F, B\}$

Free groups: universal property

Proposition

Let $\mathcal{A}$ be a set, $F_{\mathcal{A}}$ the free group on $\mathcal{A}$ and $i : A \to F_{\mathcal{A}}$ the inclusion. Let G be any group. For every $f : \mathcal{A} \to G$ there is a unique group homomorphism $g : F_{\mathcal{A}} \to G$ with $g \circ i = f$.



Equivalent moves

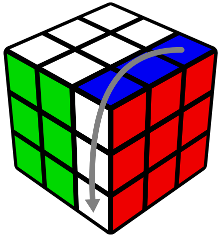
R^3

R^{-1}

R^{67}

$FBU^2D^2F^{-1}B^{-1}L^3FBU^2D^2F^{-1}B^{-1}$

All do the same to the cube

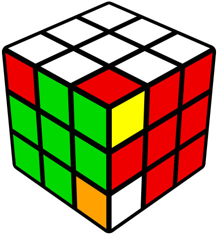


Equivalent moves

$R^{-1}D^{-1}RU^{-1}R^{-1}DRU$

$ULD^{-1}L^{-1}U^{-1}LDL^{-1}$

Which is “better”?



Equivalence relation

Call $\mathcal{S}$ the group of move sequences (free group)

Definition

For $x, y \in \mathcal{S}$ we let

$$x \approx y \iff x \text{ and } y \text{ have the same effect on the cube}$$

Equivalence relation

Call $\mathcal{S}$ the group of move sequences (free group)

Definition

For $x, y \in \mathcal{S}$ we let

$$\begin{aligned} x \approx y \quad &\iff \quad x \text{ and } y \text{ have the same effect on the cube} \\ &\iff \quad xy^{-1} \text{ does nothing to the cube} \end{aligned}$$

Group quotient

For $x \in \mathcal{S}$, let $[x] = \{y \in \mathcal{S} \mid x \approx y\}$

Define $\mathcal{M} = \{[x] \mid x \in \mathcal{S}\}$ with $[x] \times [y] = [xy]$

Question

Is $\times$ *well-defined?*

Answer

Yes:

$$x' \approx x,\ y' \approx y \implies x'y' \approx xy$$

Congruence relations vs normal subgroups

Congruence relations on G
(equivalence relations with
$x' \sim x,\ y' \sim y \implies x'y' \sim xy$)

$N = [1]$

$x \sim y \iff xy^{-1} \in N$

Normal subgroups of G
(Subgroups $N \subseteq G$ with
$n \in N,\ g \in G \implies g^{-1}ng \in N$)

Move sequences that do nothing

$\mathcal{N} = \{x \in \mathcal{S} \mid x \text{ does nothing to the cube}\}$

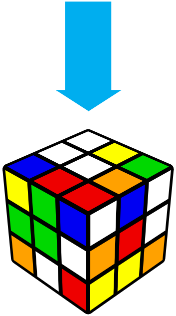
It is a subgroup:

- The empty sequence does nothing
- If x and y do nothing, so does xy
- If x does nothing, so does x^{-1}

It is normal:

- If x does nothing and y is any move sequence, $y^{-1}xy$ does the same as $y^{-1}y$, which is the empty sequence

Cube configurations



Definition

$\mathcal{C}_0 = \{$all cube configurations obtained by disassembling and re-assembling the pieces$\}$

Move sequences and configurations

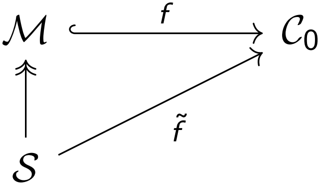
$$\mathcal{M} \xhookrightarrow{\quad f \quad} \mathcal{C}_0$$

Question

Is f surjective?

$f([x]) =$ the configuration obtained applying x to the solved cube

Move sequences and configurations



Question

Is $\tilde{f}$ surjective?

$\tilde{f}(x) =$ the configuration obtained applying x to the solved cube

Moving the cube (algebraically)

Definition

$$C_0 \times \mathcal{S} \to \mathcal{C}_0$$
$$(c, s) \mapsto c \cdot s = \text{obtained from } c \text{ applying } s$$

Examples

 $\cdot F^2 =$ 

 $\cdot RU = \Big($  $\cdot R \Big) \cdot U =$  $\cdot U =$ 

Group actions

Let G be a group, X a set

Definition

A (right) action of G on X is a map

$$\varphi : X \times G \to X$$

such that for every $g, h \in G$ and every $x \in X$

$$\varphi(x, 1) = x \qquad \text{and} \qquad \varphi(x, gh) = \varphi(\varphi(x, g), h)$$

Group actions

- Orbit: $x \cdot G = \{x \cdot g \mid g \in G\}$
- Stabilizer: $\mathrm{Stab}_G(x) = \{g \in G \mid x \cdot g = x\}$
- Associated map $G \to \mathrm{Sym}(X)$ by $g \mapsto \varphi(-, g)$
- $\ker(G \to \mathrm{Sym}(X)) = \bigcap_{x \in X} \mathrm{Stab}_G(x)$ is normal in G

$\mathcal{S}$ vs $\mathcal{M}$ revisited

For the action of $\mathcal{S}$ on $\mathcal{C}_0$, for any $c, c' \in \mathcal{C}_0$

$$\mathsf{Stab}_{\mathcal{S}}(c) = \mathsf{Stab}_{\mathcal{S}}(c') = \mathcal{N}$$

So

$$\mathcal{M} = \mathcal{S}/\mathcal{N}$$

and the *induced* action of $\mathcal{M}$ is *free*:

$$\forall\, c \in \mathcal{C}_0 : \qquad c \cdot [m] = c \quad \Longrightarrow \quad [m] = 1$$

Legal configurations

Configurations obtainable by applying moves to a solved cube

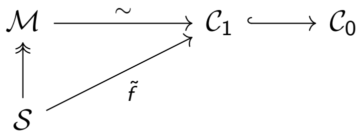
$\mathcal{C}_1 :=$  $\cdot \mathcal{M}$

form an $\mathcal{M}$-torsor ($f : \mathcal{M} \to \mathcal{C}_1$ is a bijection)

Moving vs disassembling

Question

Is the action transitive, i.e. $\mathcal{C}_1 = \mathcal{C}_0$?
If not, how many orbits are there?



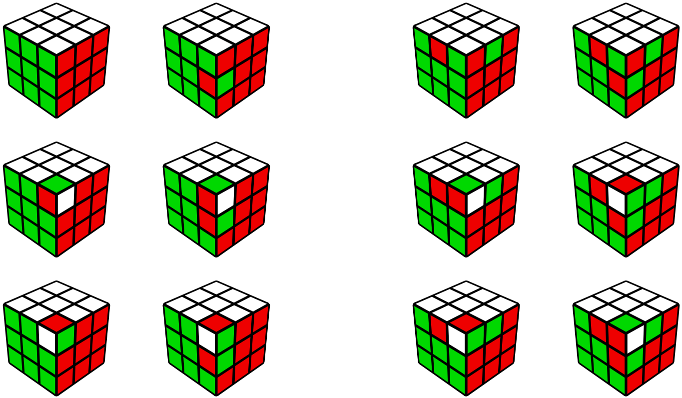
Invariants (sketch)

For every element of $\mathcal{C}_1$:

- Permutation of edges and of corners have the same sign
- The sum of “orientations” of edges is even
- The sum of “orientations” of corners is a multiple of 3

(hard to define orientation, easy to prove)

Generators for the 12 orbits



Counting configurations

- The orbits of $\mathcal{C}_0$ have the same size (because same stabilizers)
- $\#\mathcal{C}_0 = 12! \cdot 8! \cdot 2^{12} \cdot 3^8$ (simple combinatorics)
- $\#\mathcal{M} = \#\mathcal{C}_1 = \#\mathcal{C}_0/12 = 12! \cdot 8! \cdot 2^{10} \cdot 3^7$

Recap

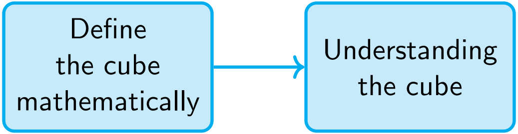
$\mathcal{C}_0$	Cube configuration (disassembly)
$\mathcal{C}_1 \subseteq \mathcal{C}_0$	Cube configuration (legal)
$\mathcal{S}$	Free group of move sequences
Action of $\mathcal{S}$ on $\mathcal{C}_0$	Moving the cube
$\mathcal{N} \subseteq \mathcal{S}$	Sequences that do nothing
$\mathcal{M} = \mathcal{S}/\mathcal{N}$	Move sequences, up to "same effect"

The pursuit of happiness

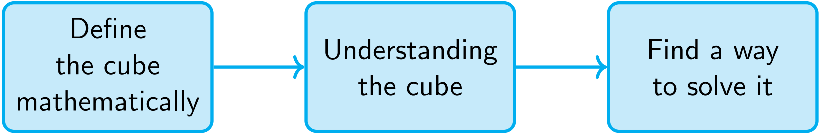
The pursuit of happiness

Define
the cube
mathematically

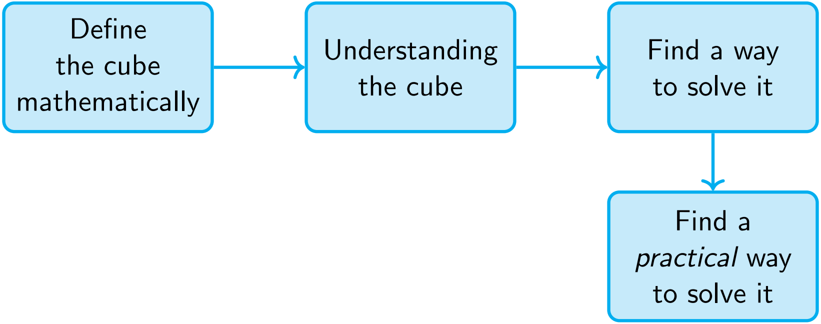
The pursuit of happiness



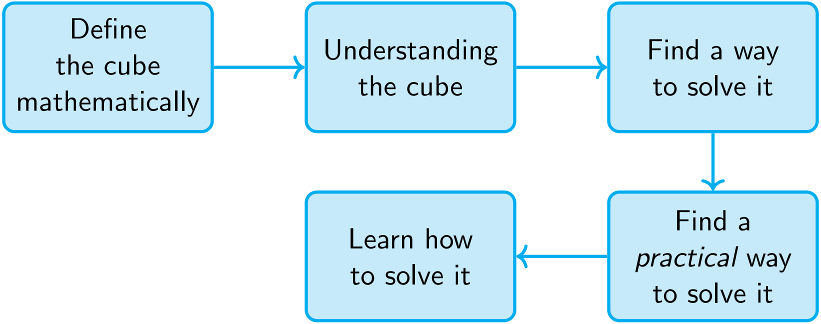
The pursuit of happiness



The pursuit of happiness



The pursuit of happiness



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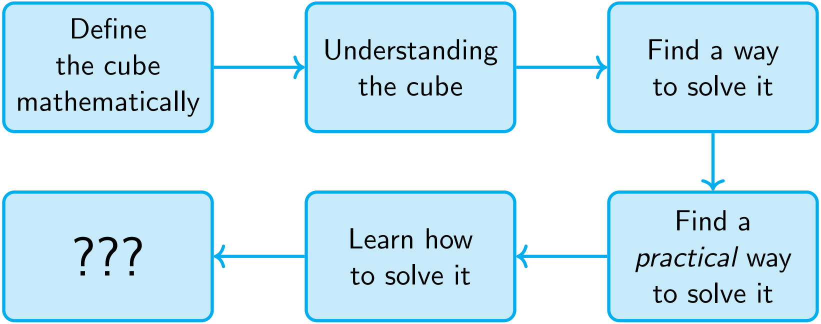

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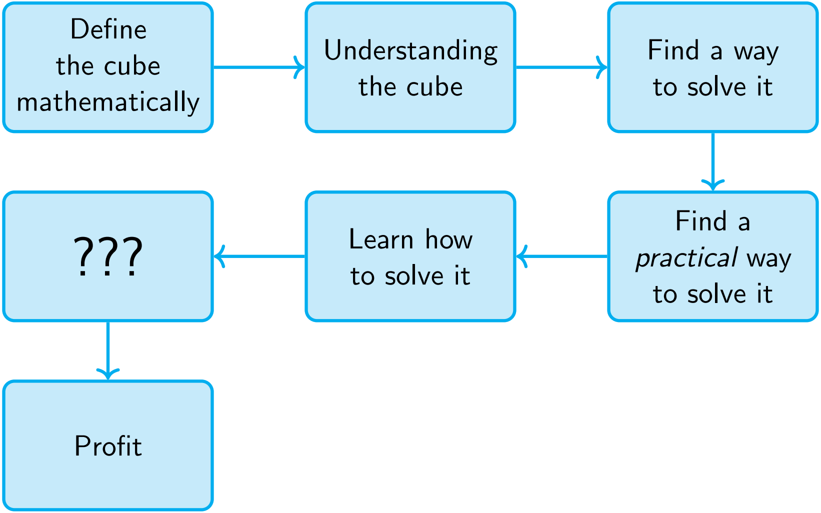

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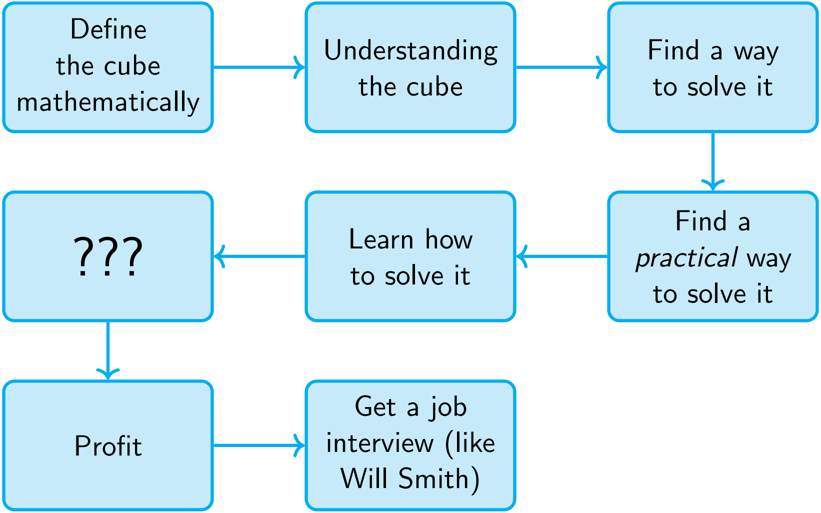

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