Field Extensions and Elliptic Curves

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- Examples: $[\mathbb{C} : \mathbb{R}] = 2$ (basis $\{1, i\}$) and $[\mathbb{R} : \mathbb{Q}] = +\infty$

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- Application: find integer solutions of $x^2 - 3y^2 = 1$

Galois Theory

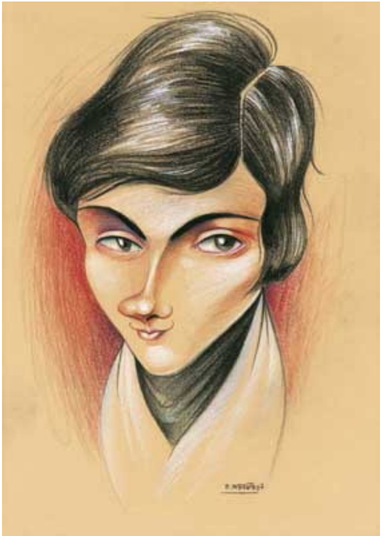

Figure: Évariste Galois

Let $L \mid K$, assume $[L : K]$ finite.

- **Separability**: $L \mid K$ is separable if the minimal polynomials of elements of L have distinct roots.
  Example: if char $K = 0$ or K is finite $L \mid K$ is separable.
- **Normality**: $L \mid K$ is normal if every irreducible $f(x) \in K[x]$ that has a root in L has **all** its roots in L.
  Example: splitting fields.

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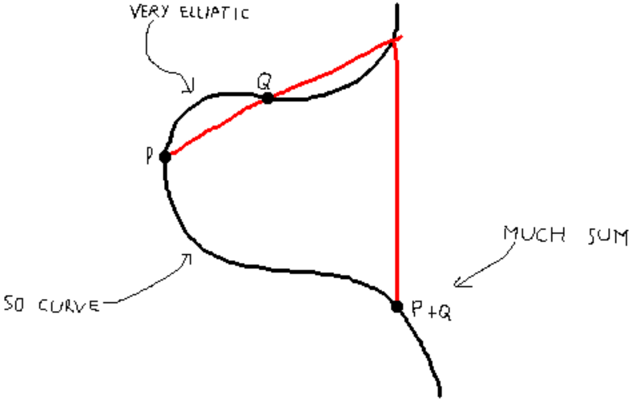
- Galois correspondence

$$\begin{aligned} \{\text{extensions } K \subseteq F \subseteq L\} &\overset{1:1}{\longleftrightarrow} \{\text{subgroups } H \leq \mathrm{Gal}(L \mid K)\} \\ F &\longmapsto \mathrm{Gal}(L \mid F) \\ \text{fixed field of } H &\longleftarrow\!\shortmid\; H \end{aligned}$$

Elliptic Curves



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- Applications in Cryptography

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- Nicer definition: $K(P_1, \dots, P_n)$ is the subfield of $\overline{K}$ fixed by

$$\left\{\sigma \in \mathrm{Gal}(\overline{K} \mid K) \mid \sigma(P_i) = P_i \text{ for } i = 1, \dots, n\right\}$$

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- The action of $\mathrm{Gal}(K(E[n]) \mid K)$ on $E[n] \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ gives a **Galois representation**

$$\rho_n : \mathrm{Gal}(K(E[n]) \mid K) \hookrightarrow \mathrm{Aut}(E[n]) \cong \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

Kummer Theory

Figure: Ernst Kummer

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- One can e.g. compute the degree $[L : K(\zeta_n)]$.

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Theorem (joint with Davide Lombardo)

There is an explicit constant C, depending only on P and on the torsion Galois representations associated to E such that

$$\frac{n^2}{[K(n^{-1}P) : K(E[n])]} \quad \textit{divides} \quad C$$

for all $n \geq 1$.

Thank you for your attention