Kummer theory for algebraic groups

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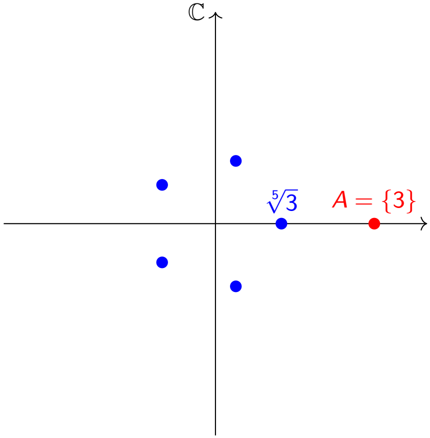
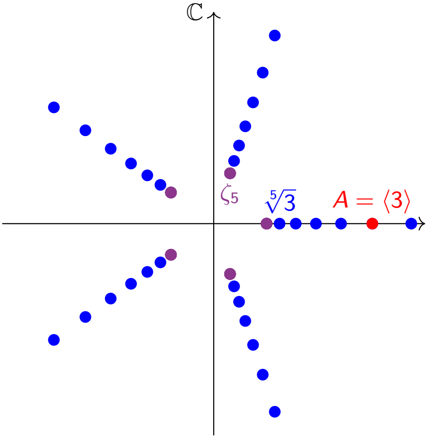
Kummer theory



- $A \subseteq \mathbb{Q}^\times$, $\sqrt[n]{A} = \{x \in \mathbb{C} \mid x^n \in A\}$
- *Kummer extension* $\mathbb{Q}(\sqrt[n]{A})$
- Galois over $\mathbb{Q}$, contains $\mathbb{Q}(\zeta_n)$

Kummer theory



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Kummer theory for algebraic groups

G commutative algebraic group over K number field

- $A \leq G(K)$, $n^{-1}A = \{P \in G(\overline{K}) \mid nP \in A\}$
- "Kummer extension" $K(n^{-1}A)$
- Galois over K, contains $K(G(\overline{K})[n])$
- Classical Kummer theory when $G = \mathbb{G}_m$

Results for elliptic curves

$G = E$ elliptic curve, $A = \langle \alpha \rangle$

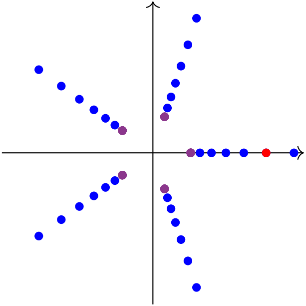
- Ribet, 1979: $cn^2 \leq [K(n^{-1}A) : K(E(\overline{K})[n])] \leq n^2$
- Lombardo-T., 2020: Effective $c = c(E, K, \alpha)$ if no CM
- Lombardo-T., 2021: over $K = \mathbb{Q}$

  $$c^{-1} \leq 2^{28} \cdot 3^{18} \cdot 5^8 \cdot 7^7 \cdot 11^5 \cdot 13 \cdot 17 \cdot 19 \cdot 37 \cdot 43 \cdot 67 \cdot 163$$

- A. Javan Peykar, 2021: CM case

Endomorphism rings

A. Javan Peykar, 2021: CM case $\rightarrow$ take A an $\mathrm{End}_K(E)$-module



Over $\mathbb{Z}$

$\rightsquigarrow$

?

Over $\mathrm{End}_K(E)$

Division modules

R ring, $M \subseteq N$ (left) modules, I (right) ideal

$$(M :_N I) := \{x \in N \mid Ix \subseteq M\}$$

Considering certain families (filters) $\mathcal{J}$ of ideals

$$(M :_N \mathcal{J}) := \bigcup_{I \in \mathcal{J}} (M :_N I)$$

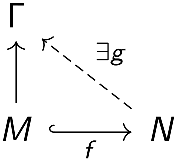
Ideal filters

Examples

$$\infty := \{I \textit{ ideal of } R \mid I \supseteq nR \textit{ for some } n \geq 1\}$$

$$\mathfrak{p}^\infty := \{I \textit{ ideal of } R \mid I \supseteq p^k R \textit{ for some } k \geq 0\} \quad (p \textit{ prime})$$

$\mathcal{J}$-injectivity



$(f(M) :_N \mathcal{J}) = N$

Γ is $\mathcal{J}$-injective if maps to Γ lift over "$\mathcal{J}$-extensions"

Remark

- *Injective* $\iff$ *$\mathcal{J}$-injective for $\mathcal{J} = \{$all ideals$\}$*
- *Over $\mathbb{Z}$: p-divisible* $\iff$ *$\mathfrak{p}^\infty$-injective*

$(\mathcal{J}, T)$-extensions

Fix a $\mathcal{J}$-injective module $T = T[\mathcal{J}] \leftrightsquigarrow G(\overline{K})_{\text{tors}}$

- $M \subseteq N$ with $(M :_N \mathcal{J}) = N$ and $N[\mathcal{J}] \hookrightarrow T$
- Galois-like category
- "$\mathcal{J}$-injective hull" $\leftrightsquigarrow \bigcup_{n \geq 1} n^{-1} A$

Galois representations

$$A \leq G(K) \qquad \Gamma = \bigcup_{n\geq 1} n^{-1}A \subseteq G(\overline{K}) \qquad T = G(\overline{K})_{\mathrm{tors}}$$

$$\begin{array}{c} \mathrm{Gal}(K(\Gamma) \mid K) \\ \big\downarrow \\ \mathrm{Aut}_A(\Gamma) \end{array}$$

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$$\begin{array}{ccccc} & & \mathrm{Gal}(K(\Gamma) \mid K) & & \\ & & \big\downarrow & & \\ \mathrm{Aut}_{A+T}(\Gamma) & \hookrightarrow & \mathrm{Aut}_A(\Gamma) & \twoheadrightarrow & \mathrm{Aut}_{A_{\mathrm{tors}}}(T) \end{array}$$

Galois representations

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\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{Aut}_{A+T}(\Gamma) & \hookrightarrow & \mathrm{Aut}_{A}(\Gamma) & \twoheadrightarrow & \mathrm{Aut}_{A_{\text{tors}}}(T)
\end{array}$$

New results

- Completed and unified CM and non-CM cases
- Better understanding of Kummer theory for algebraic groups
- In progress: higher-dimensional abelian varieties

Thank you for your attention